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Additive mappings that preserve rank one nilpotent operators

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Abstract

Let $B(X)$ be the algebra of bounded operators on a complex infinite dimensional Banach space X, and $F(X)$ be the subalgebra of all finite rank operators in $B(X)$. A characterization of additive mappings on $F(X)$ which preserve rank one nilpotent operators in both directions is given. As applications of this result, some additive preservers are described.

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1. Introduction

One of the most active and fertile subjects in matrix theory during the past 100 years is the linear preserver problem, which concerns the characterization of linear operators on matrix space that leave certain functions, subsets, relations, etc., invariant (see the survey paper [6]). It seems that in the last decades there has been a considerable interest in analogous problems for operator algebras over infinite dimensional spaces (see the survey paper [2]).

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When discussing a preserver on an operator algebra one usually assumes that this mapping is linear. A more general approach would be to consider this algebra only as a ring, therefore, the preserver would be additive. In this direction, the first result was obtained in [9] which characterized the rank one preserving additive mapping. It is the aim of this note to continue the study of additive preservers by considering additive mappings that preserve rank one nilpotent operators in both directions. Using this result we also describe some additive preservers. It is interesting that some linear preserver results can be extended to additive preserver results by slight modification of the proofs as shown in this paper.

It should be mentioned here that additive rank one nilpotence preservers play an important role in additive preserver problems, because in many cases additive preservers can be reduced to the problems of rank one nilpotence preservers and rank one idempotence preservers, while the form of rank one nilpotence preserver is the key to characterize the rank one idempotence preserver.

Before we proceed let us fix some notation. Let $B(X)$ be the Banach algebra of bounded linear oprators on a Banach space X. An operator $T \in B(X)$ is said to be potent (resp. nilpotent) if there exists an integer $r \geqslant 2$ such that $T^r = T$ (resp. $T^r = 0$). Particularly, if $T^2 = T$, then T is said to be idempotent. We will denote by $x \otimes f$ the bounded linear operator on X defined for any $x \in X$ and $f \in X^*$, the dual space of X, by $(x \otimes f)y = f(y)x$ for arbitrary $y \in X$. For any Banach space X we denote by $F(X)$, $N(X)$, $N_k(X)$ and $B_0(X)$ the set of all finite rank linear operators, the set of all nilpotent linear bounded operators, the set of all nilpotent linear bounded operators with nilindex no greater than k, and the linear span of the set of all nilpotent operators in $B(X)$, respectively.

Let τ be a ring automorphism of $\mathbf{C}$, the complex number field. A mapping $A : X \to X$ will be called τ-quasilinear if it is additive and if the relation $A(\lambda x) = \tau(\lambda)Ax$ holds for all complex number λ and all $x \in X$. Particularly, A is said to be conjugate-linear if it is additive and $A(\lambda x) = \bar{\lambda}Ax$ for all $x \in X$ and $\lambda \in \mathbf{C}$. If A is conjugate-linear, we will define A^* by $(A^*f)(x) = \overline{f(Ax)}$ for all $x \in X$ and $f \in X^*$.

Throughout this paper all Banach spaces X and Hilbert spaces H will be infinite dimensional.

2. Additive mappings preserving rank one nilpotent (resp. idempotent) operators

In this section, mapping $\phi : F(X) \to F(X)$ will be an additive surjective mapping preserving rank one nilpotent operators in both directions, that is, $\phi(T)$ is a rank one nilpotent operator if and only if T is a nilpotent operator of rank 1.

For each nonzero $x \in X$ and $f \in X^*$, let $\{x\}^\perp = \{g \in X^* : g(x) = 0\}$, $\{f\}^\perp = \{y \in X : f(y) = 0\}$, $L_x^\perp = \{x \otimes g : g \in \{x\}^\perp\}$ and $R_f^\perp = \{y \otimes f : y \in \{f\}^\perp\}$.

To prove the main result, we need some lemmas. Let us begin with

Lemma 2.1

(1) *For each* $x \in X$, *either there is a* $y \in X$ *such that* $\phi(L_x^\perp) = L_y^\perp$ *or there is a* $g \in X^*$ *such that* $\phi(L_x^\perp) = R_g^\perp$.

(2) *For each* $f \in X^*$, *either there is a* $u \in X$ *such that* $\phi(R_f^\perp) = L_u^\perp$ *or there is an* $h \in X^*$ *such that* $\phi(R_f^\perp) = R_h^\perp$.

Proof. We only prove (1). And (2) can be proved similarly.

Fix a nonzero $x \in X$, and choose two arbitrary functionals $f_1, f_2 \in \{x\}^\perp$ such that $f_1 + f_2 \neq 0$. Suppose that $\phi(x \otimes f_1) = u_1 \otimes h_1$ and $\phi(x \otimes f_2) = u_2 \otimes h_2$. Obviously, $h_1(u_1) = h_2(u_2) = 0$. Then $u_1 \otimes h_1 + u_2 \otimes h_2$ as well as $x \otimes f_1 + x \otimes f_2$ is a rank one nilpotent operator, this implies that either u_1 and u_2 are linearly dependent or h_1 and h_2 are linearly dependent. Then let $y = u_1$ and $g = h_1$ respectively, and so we have either $\phi(L_x^\perp) \subseteq L_y^\perp$ or $\phi(L_x^\perp) \subseteq R_g^\perp$.

With no loss of generality, suppose $\phi(L_x^\perp) \subseteq L_y^\perp$. We shall show $\phi(L_x^\perp) = L_y^\perp$.

Assume to the contrary that there exists $g_0 \in \{y\}^\perp$ such that $y \otimes g_0 \notin \phi(L_x^\perp)$. Suppose $\phi(x_0 \otimes f_0) = y \otimes g_0$ for some $x_0 \in X$ and $f_0 \in X^*$. Obviously x_0 and x are linearly independent. Choose two linearly independent functionals f_1 and f_2 in $\{x\}^\perp$ such that $g_1 \neq -g_0 \neq g_2$, where $\phi(x \otimes f_1) = y \otimes g_1$ and $\phi(x \otimes f_2) = y \otimes g_2$. Since both $y \otimes g_0 + y \otimes g_1$ and $y \otimes g_0 + y \otimes g_2$ are nilpotents of rank 1, we can infer that both $x_0 \otimes f_0 + x \otimes f_1$ and $x_0 \otimes f_0 + x \otimes f_2$ are also nilpotents of rank 1. But x_0 and x are linearly independent, thus f_0 and f_1 as well as f_0 and f_2 are linear dependent, which leads to a contradiction with the fact that f_1 and f_2 are linearly independent. □

Lemma 2.2

(1) $\phi(L_x^\perp) = L_y^\perp$ *and* $\phi(L_z^\perp) = R_h^\perp$ *cannot hold simultaneously*;

(2) $\phi(R_f^\perp) = L_u^\perp$ *and* $\phi(R_g^\perp) = R_k^\perp$ *cannot hold simultaneously.*

Proof. We only prove (1), and (2) goes similarly.

Assume to the contrary that $\phi(L_x^\perp) = L_y^\perp$ and $\phi(L_z^\perp) = R_h^\perp$. Then by Lemma 2.1, x and z must be linearly independent, and we have $h(y) \neq 0$. Choose a nonzero $f \in \{x, z\}^\perp$ and let $\phi(x \otimes f) = y \otimes k$ and $\phi(z \otimes f) = u \otimes h$, then obviously $k(y) = h(u) = 0$. It follows that $y \otimes k + u \otimes h$ is a rank one nilpotent operator as $x \otimes f + z \otimes f$ is a nilpotent operator of rank 1. But on the other hand, from $h(y) \neq 0$ and $k(y) = h(u) = 0$ we can infer that u and y as well as h and k are linearly independent, hence $y \otimes k + u \otimes h$ is of rank 2, this contradiction completes the proof. □

Lemma 2.3

(1) *If for each* $x \in X$ *there exists a* $y \in X$ *such that* $\phi(L_x^\perp) = L_y^\perp$, *then for each* $f \in X^*$ *there exists a* $g \in X^*$ *such that* $\phi(R_f^\perp) = R_g^\perp$;

(2) *If for each* $x \in X$ *there exists a* $g \in X^*$ *such that* $\phi(L_x^{\perp}) = R_g^{\perp}$, *then for each* $f \in X^*$ *there exists a* $y \in X$ *such that* $\phi(R_f^{\perp}) = L_y^{\perp}$.

Proof. We prove (1) only.

Suppose that for each $x \in X$ there exists a $y \in X$ such that $\phi(L_x^{\perp}) = L_y^{\perp}$. Assume to the contrary that there exist an $f \in X^*$ and a $z \in X$ such that $\phi(R_f^{\perp}) = L_z^{\perp}$. We have two cases:

Case 1. $f(x) = 0$.

Obviously, $\phi(x \otimes f) \in L_y^{\perp} \cap L_z^{\perp}$, hence y and z are linearly dependent. Suppose that $\phi(x \otimes f) = z \otimes h$. Choose $u \in \{f\}^{\perp}$ such that u and x are linearly independent. Let $\phi(L_u^{\perp}) = L_v^{\perp}$. It is easy to see that v and z are linearly dependent, let $v = \lambda z$ for some $\lambda \in \mathbf{C}$. We can find $g \in \{u\}^{\perp}$ such that g and f are linearly independent, and let $\phi(u \otimes g) = v \otimes k$. Then we have $\phi(x \otimes f + u \otimes g) = z \otimes h + v \otimes k = z \otimes (h + \lambda k)$. If $h + \lambda k \neq 0$, then $z \otimes h + v \otimes k$ is a rank one nilpotent, but $x \otimes f + u \otimes g$ is of rank 2, which contradicts the propterty of ϕ. If $h + \lambda k = 0$, then $\phi(x \otimes f + u \otimes 2g) = z \otimes (h + 2\lambda k)$ is a nilpotent of rank 1, hence $x \otimes f + u \otimes 2g$ must be a rank one nilpotent, this contradicts the fact that $x \otimes f + u \otimes 2g$ is of rank 2.

Case 2. $f(x) \neq 0$.

If y and z are linearly dependent, then we can find $u \in \{f\}^{\perp}$ and $g \in \{x\}^{\perp}$ but $g \notin \{u\}^{\perp}$. Suppose $\phi(x \otimes g) = y \otimes k$ and $\phi(u \otimes f) = z \otimes h$. Obviously, $k(y) = h(z) = 0$. Since y and z are linearly dependent, by a similar argument as above, we can suppose that $y \otimes k + z \otimes h$ is a rank one nilpotent operator, hence $x \otimes g + u \otimes f$ is also a nilpotent operator of rank 1. But from the choice of u and g, we know that $x \otimes g + u \otimes f$ is of rank 2, a contradiction.

Let now y and z be linearly independent. Choose a nonzero $g \in \{y, z\}^{\perp}$, and let $\phi(x \otimes h) = y \otimes g$ and $\phi(u \otimes f) = z \otimes g$. It is obvious that $g(y) = g(z) = h(x) = f(u) = 0$. Since $f(x) \neq 0$ and $f(u) = 0$, we know that x and u are linearly independent. Similarly, we can infer that f and h are also linearly independent. Hence $x \otimes h + u \otimes f$ is of rank 2. But on the other hand, $x \otimes h + u \otimes f$ is of rank 1 since $y \otimes g + z \otimes g$ is a rank one nilpotent operator, a contradiction. □

Now we are ready to prove our main result.

Theorem 2.4. *Let* X *be a complex infinite dimensional Banach space*, $\phi : F(X) \to F(X)$ *be an additive surjective mapping preserving rank one nilpotent operators in both directions*, *then there exists a nonzero complex number* c *such that either*

(1) *there is a bounded linear or conjugate-linear bijective mapping* $A : X \to X$ *such that* $\phi(x \otimes f) = cAx \otimes fA^{-1}$ *holds for all* $x \in X$ *and* $f \in X^*$ *with* $f(x) = 0$; *or*

(2) *there is a bounded linear or conjugate-linear bijective mapping* $A : X^* \to X$ *such that* $\phi(x \otimes f) = cA(x \otimes f)^* A^{-1}$ *holds for all* $x \in X$ *and* $f \in X^*$ *with* $f(x) = 0$. *In this case*, X *must be reflexive.*

Proof. For each $x \in X$ we have two cases:

Case 1. there exists a $y \in X$ such that $\phi(L_x^{\perp}) = L_y^{\perp}$.

By a proof similar to that of [11] we can find two additive mappings $A : X \to X$ and $C : X^* \to X^*$ such that

$$\phi(x \otimes f) = Ax \otimes Cf$$

holds for all $x \in X$ and $f \in X^*$ with $f(x) = 0$.

We claim that both A and C are bijective. Note that the surjectivity of A and C follows from the surjectivity of ϕ and the injectivity from the fact that ϕ preserves rank one nilpotents in both directions.

Using the methods from the proof of the Main Theorem in [9] one can show that there exists a continuous ring automorphism $\tau : \mathbf{C} \to \mathbf{C}$ such that both A and C are τ-quasilinear. Therefore, both A and C are either linear or conjugate-linear simultaneously.

With no loss of generality, suppose $\tau(\lambda) = \lambda$. Then both A and C are linear, and we have $(Cf)(Ax) = cf(X)$ for all $x \in X$ and $f \in X^*$. Now the boundedness of C and A follows by applying the Closed Graph Theorem directly to the proceeding observation, $(Cf)(Ax) = cf(X)$. Thus we have

$$\phi(x \otimes f) = Ax \otimes fA^{-1}$$

for all $x \in X$ and $f \in X^*$ with $f(x) = 0$.

Case 2. there exists an $f \in X^*$ such that $\phi(L_x^{\perp}) = R_f^{\perp}$.

By a proof similar to the above and the linear case (see [11]), one can complete the proof. □

Now we turn our attention to the rank one idempotent preservers. In the rest of this section, mapping $\psi : F(X) \to F(X)$ will be an additive bijection preserving rank one idempotents in both directions.

For arbitrary nonzero $x \in X$ and $f \in X^*$, let $\{x\}^1 = \{g \in X^* : g(x) = 1\}$, $\{f\}^1 = \{y \in X : f(y) = 1\}$, $L_x^1 = \{x \otimes g : g \in \{x\}^1\}$, and $R_f^1 = \{y \otimes f : y \in \{f\}^1\}$.

Lemma 2.5

(1) *For each* $x \in X$, *either there is a* $y \in X$ *such that* $\psi(L_x^1) = L_y^1$ *or there is a* $g \in X^*$ *such that* $\psi(L_x^1) = R_g^1$;

(2) *For each* $f \in X^*$, *either there exists a* $u \in X$ *such that* $\psi(R_f^1) = L_u^1$ *or there exists an* $h \in X^*$ *such that* $\psi(R_f^1) = R_h^1$.

Lemma 2.6
(1) $\psi(L_x^1) = L_z^1$ *and* $\psi(L_y^1) = R_h^1$ *cannot hold simultaneously*;
(2) $\psi(R_f^1) = L_u^1$ *and* $\psi(R_g^1) = R_k^1$ *cannot hold simultaneously.*

Lemma 2.7
(1) *If for each* $x \in X$ *there is a* $y \in X$ *such that* $\psi(L_x^1) = L_y^1$, *then for each* $f \in X^*$ *there must be a* $g \in X^*$ *such that* $\psi(R_f^1) = R_g^1$;
(2) *If for each* $x \in X$ *there is an* $f \in X^*$ *such that* $\psi(L_x^1) = R_f^1$, *then for each* $g \in X^*$ *there must be a* $y \in X$ *such that* $\psi(R_g^1) = L_y^1$.

The proofs of the above three lemmas are similar to those of Lemmas 2.1–2.3, so we omit them.

Lemma 2.8. ψ *preserves rank one nilpotents in both directions.*

Proof. Let $x \otimes f$ be any rank one nilpotent. With no loss of generality, suppose $\psi(L_x^1) = L_y^1$ for some $y \in X$. Choose $g \in \{x\}^1$, and let $\psi(x \otimes g) = y \otimes h$, $\psi(x \otimes (f+g)) = y \otimes k$, then $h(y) = k(y) = 1$. On the other hand, $\psi(x \otimes (f+g)) = \psi(x \otimes f) + \psi(x \otimes g)$, hence $y \otimes k = \psi(x \otimes f) + y \otimes h$, and so $\psi(x \otimes f) = y \otimes (k-h)$ and $(k-h)(y) = 0$. Since ψ is injective, $y \otimes (k-h)$ is a rank one nilpotent. Thus we can infer that ψ preserves rank one nilpotents. Similarly, ψ^{-1} also preserves rank one nilpotents. And so ψ preserves rank one nilpotents in both directions. $\square$

Lemma 2.9
(1) *If for each* $x \in X$ *there exists a* $y \in X$ *such that* $\psi(L_x^1) = L_y^1$, *then we must have* $\psi(L_x^\perp) = L_y^\perp$;
(2) *If for each* $x \in X$ *there exists a* $g \in X^*$ *such that* $\psi(L_x^1) = R_g^1$, *then we must have* $\psi(L_x^\perp) = R_g^\perp$;
(3) *If for each* $f \in X^*$ *there exists a* $u \in X$ *such that* $\psi(R_f^1) = L_u^1$, *then we must have* $\psi(R_f^\perp) = L_u^\perp$;
(4) *If for each* $f \in X^*$ *there exists an* $h \in X^*$ *such that* $\psi(R_f^1) = R_h^1$, *then we must have* $\psi(R_f^\perp) = R_h^\perp$.

Proof. We only prove (1), similarly we can prove (2)–(4).

Suppose that $\psi(L_x^1) = L_y^1$. For each $f \in \{x\}^1$, let $\psi(x \otimes f) = y \otimes g$. And for any $h \in \{x\}^\perp$, by Lemma 2.8, let $\psi(x \otimes h) = z \otimes k$ with $k(z) = 0$. Observe that $x \otimes (f+h)$ is an idempotent of rank 1, suppose $\psi(x \otimes (f+h)) = y \otimes l$ with $l(y) = 1$, then we have $y \otimes l = y \otimes g + z \otimes k$, this yields that z and y are linearly dependent, let $z = \alpha y$ for some complex number α. Then $\psi(x \otimes h) = y \otimes \alpha k$, this leads to $\psi(L_x^\perp) \subseteq L_y^\perp$. By a similar argument as in the proof of Lemma 2.1, one can easily have $\psi(L_x^\perp) = L_y^\perp$. $\square$

Now we are in a position to characterise the additive rank one idempotent preservers.

Theorem 2.10. *Let X be an infinite dimensional Banach space. Let $\psi : F(X) \to F(X)$ be an additive bijective mapping preserving rank one idempotents in both directions, then either*

(1) *there exists a bounded linear or conjugate-linear bijective mapping $A : X \to X$ such that $\psi(x \otimes f) = Ax \otimes fA^{-1}$ holds for any $x \in X$ and $f \in X^*$ with $f(X)$ being rational*; *or*
(2) *there exists a bounded linear or conjugate-linear bijective mapping $A : X^* \to X$ such that $\psi(x \otimes f) = A(x \otimes f)^* A^{-1}$ holds for any $x \in X$ and $f \in X^*$ with $f(X)$ being rational. In this case, X must be reflexive.*

Proof. For each $x \in X$ we have two cases:

Case 1. $\psi(L_x^1) = L_y^1$.

By Lemma 2.9, we have $\psi(L_x^\perp) = L_y^\perp$, thus by Theorem 2.4 there exist a complex number c and a bounded linear or conjugate-linear bijective mapping $A : X \to X$ such that $\psi(x \otimes f) = cAx \otimes (A^{-1})^* f$ for all $x \in X$ and $f \in X^*$ with $f(x) = 0$.

Without loss of generality we may assume that A is linear. For arbitrary rank one idempotent $x \otimes f$, suppose $\psi(x \otimes f) = y \otimes h$. Choose an arbitrary $g \in \{x\}^\perp$, then $\psi(x \otimes g) = cAx \otimes (A^{-1})^* g$. Since $x \otimes (f + g)$ is an idempotent of rank 1, let $\psi(x \otimes (f + g)) = y \otimes k$, then $y \otimes k = cAx \otimes (A^{-1})^* g + y \otimes h$, therefore y and Ax are linearly dependent, so we can write $y = \alpha Ax$ for some complex number α. Similarly, we can prove that h and $(A^{-1})^* f$ are also linearly dependent, write $h = \beta(A^{-1})^* f$ for some complex number β.

Thus for arbitrary rank one idempotent $x \otimes f$ we have

$$\psi(x \otimes f) = \alpha\beta Ax \otimes (A^{-1})^* f.$$

Since ψ preserves rank one idempotents in both directions, it follows that $\alpha\beta = 1$ and so

$$\psi(x \otimes f) = Ax \otimes fA^{-1}$$

holds for all $x \in X$ and $f \in X^*$ with $f(x) = 1$.

Since every rank one nilpotent can be written as a difference of two rank one idempotents, it is easy to see that the above equality holds for all $x \in X$ and $f \in X^*$ with $f(x) = 0$. Moreover, as ψ is additive, it follows that

$$\psi(x \otimes f) = Ax \otimes fA^{-1}$$

holds for all $x \in X$ and $f \in X^*$ with $f(X)$ being rational.

Case 2. $\psi(L_x^1) = R_f^1$.

The proof of this case is similar to those of Case 1 and Theorem 2.4, so we omit it. □

3. Applications

We first classify some additive preservers which can be reduced to the problem of rank one nilpotent preservers.

We begin with nilpotent additive preservers. Recall that mapping $\phi : B(X) \to B(X)$ is said to be preserving nilpotent operators in both directions provided $\phi(T)$ is nilpotent if and only if T is nilpotent for all $T \in B(X)$.

In order to study additive mappings preserving nilpotent operators in both directions, we need the following lemma.

Lemma 3.1. *For nonzero $N \in N(X)$, the following statements are equivalent:*

(1) *N is of rank* 1;
(2) *For every $A \in N(X)$ satisfying $A + N \notin N(X)$ we have $A + \lambda N \notin N(X)$ for all nonzero $\lambda \in \mathbf{C}$*;
(3) *For every $A \in N(X)$ satisfying $A + N \notin N(X)$ we have $A + 2N \notin N(X)$.*

Proof. See the proof of Proposition 2.1 in [11]. □

Now we are in a position to prove the following.

Theorem 3.2. *Let X be a complex infinite dimensional Banach space, $\phi : B_0(X) \to B_0(X)$ be an additive surjective mapping preserving nilpotent operators in both directions. Then there exists a nonzero $c \in \mathbf{C}$ such that either*

(1) *there is a bounded linear or conjugate-linear bijective mapping $A : X \to X$ such that $\phi(T) = cATA^{-1}$ holds for all $T \in B_0(X)$; or*
(2) *there is a bounded linear or conjugate-linear bijective mapping $A : X^* \to X$ such that $\phi(T) = cAT^*A^{-1}$ holds for all $T \in B_0(X)$. In this case, X must be reflexive.*

Proof. With the same argument as in [11], one can see that ϕ is injective. By Lemma 3.1 and the bijectivity of ϕ we know that ϕ preserves rank one nilpotent operators in both directions. Now the rest of proof is similar to the linear case (see [11]), so we omit it. □

Recall that if H is an infinite dimensional Hilbert space, then every operator of $B(H)$ can be written as a sum of five square-zero operators (see [10]). Thus $B_0(H) = B(H)$, therefore we have the following theorem.

Theorem 3.3. *Let H be a complex infinite dimensional Hilbert space, $\psi : B(H) \to B(H)$ be an additive surjective mapping preserving nilpotent operators in both directions. Then there exist a nonzero $c \in \mathbf{C}$ and a bounded linear or conjugate-linear bijective mapping $A : H \to H$ such that ψ is either of the form*

$$\psi(T) = cATA^{-1} \quad (T \in B(H))$$

or of the form

$$\psi(T) = cAT^{\mathrm{tr}}A^{-1} \quad (T \in B(H)),$$

where T^{tr} denotes the transpose of T relative to a fixed but arbitrary orthonormal basis.

Next we shall consider the mapping $\psi : B(H) \to B(H)$ satisfying $\psi(T)^k = 0$ if and only if $T^k = 0$ for all $T \in B(H)$, where k is a positive number larger than 2.

Let us first introduce a key lemma.

Lemma 3.4. *Let $k > 2$ be a positive integer, H a (finite or infinite dimensional) Hilbert space, and let $A \in N_k(H)$ be a nonzero operator. Then the following conditions are equivalent:*

(1) *A is of rank* 1;
(2) *For every $B \in N_k(H)$ satisfying $A + B \notin N_k(H)$ we have $B + \alpha A \notin N_k(H)$ for every nonzero $\alpha \in \mathbf{C}$; and*
(3) *For every $B \in N_k(H)$ satisfying $A + B \notin N_k(H)$ we have $B + 2A \notin N_k(H)$.*

Proof. See the proof of Lemma 2.2 in [12]. □

We conclude the first part of this section by the following.

Theorem 3.5. *Let H be a complex infinite dimensional Hilbert space, k be an integer no smaller than* 3, *and let $\psi : B(H) \to B(H)$ be an additive surjective mapping satisfying $\psi(T)^k = 0$ if and only if $T^k = 0$ for all $T \in B(H)$. Then there exist a nonzero $c \in \mathbf{C}$ and a bounded linear or conjugate-linear bijective mapping $A : H \to H$ such that for every $T \in B(H)$ either $\psi(T) = cATA^{-1}$ or $\psi(T) = cAT^{\mathrm{tr}}A^{-1}$.*

Proof. It follows from Lemma 3.4 that ψ preserves nilpotent operators of rank one in both directions. Then, applying Theorem 2.4 and the argument in [12], one can easily complete the proof. □

We now turn our attention to additive preservers which can be reduced to the problem of rank one idempotent additive preservers.

The following result can be followed from Corollary 3.3 in [5], here we give a different proof using rank one idempotent preservers.

Theorem 3.6. *Let H be a complex infinite dimensional Hilbert space, ψ be an additive surjective mapping preserving idempotents in both directions, then there exists a bounded linear or conjugate-linear bijective mapping $A : H \to H$ such that either $\psi(T) = ATA^{-1}$ or $\psi(T) = AT^{\mathrm{tr}}A^{-1}$ holds for all $T \in B(H)$.*

Proof. We first show that ψ is injective.

Assume that there exists a nonzero $T \in B(H)$ such that $\psi(T) = 0$, then $T^2 = T$ since $\psi(T)^2 = \psi(T)$. But we also have $(2T)^2 = 2T$ as $\psi(2T) = 0 = \psi(2T)^2$, which yields that $2T = T$, this contradiction shows the injectivity of ψ.

We claim that ψ preserves rank one idempotents in both directions.

Indeed, let $P \in B(H)$ be a rank one idempotent. If the rank of $\psi(P)$ is larger than 1, then there exist two idempotents Q_1 and Q_2 such that $\psi(P) = Q_1 + Q_2$. Since ψ preserves idempotents in both directions, there exist idempotents P_1 and P_2 such that $\psi(P_i) = Q_i$, $i = 1, 2$. Thus $P = P_1 + P_2$, a contradiction.

Hence we can conclude that ψ preserves rank one idempotents, but this holds true for ψ^{-1}, and so ψ preserves rank one idempotents in both directions.

Therefore we can apply Theorem 2.10, suppose that $\psi(E) = AEA^{-1}$ for all rank one idempotents $E \in B(H)$, where $A : H \to H$ is a bounded linear or conjugate-linear bijective mapping.

Since every operator in $B(H)$ can be written as a sum of five idempotents (see [10]), to prove $\psi(T) = ATA^{-1}$ holds for all $T \in B(H)$ it suffices to show that $\psi(P) = APA^{-1}$ holds for every idempotent $P \in B(H)$. This can be done following the closing of the proof of Theorem 1 in [3]. □

Now we consider additive potence preservers.

Theorem 3.7. *Let H be a complex infinite dimensional Hilbert space. Let ϕ : $B(H) \to B(H)$ be a unital additive surjective mapping preserving potent operators in both directions. Then there exists a bounded linear or conjugate-linear bijective mapping $A : H \to H$ such that for every $T \in B(H)$ either $\phi(T) = ATA^{-1}$ or $\phi(T) = AT^{\mathrm{tr}}A^{-1}$.*

Proof. First observe that ϕ is injective. Indeed, if there exists an $A \neq 0$ such that $\phi(A) = 0$ then A is potent and there exists a potent B such that $A + B$ is not potent. Then $\phi(A + B) = \phi(B)$ is not potent, a contradiction.

We next claim that ϕ preserves idempotents in both directions.

For arbitrary idempotent $P \in B(H)$, it is obvious that $I - P$ is idempotent, and $(I - 2P)^3 = I - 2P$. Hence $\phi(P)$, $I - \phi(P)$ and $I - 2\phi(P)$ are all potent. Suppose there exist natural numbers $r, s, t > 1$ such that $\phi(P)^r = \phi(P)$, $(I - \phi(P))^s = I - \phi(P)$ and $(I - 2\phi(P))^t = I - 2\phi(P)$. Since $\phi(P)$ is a potent operator, then $\phi(P)$ can be expressed as

$$\phi(P) = \sum_{i=1}^{k} \mu_i Q_i,$$

where $\mu_i \in \mathbf{C}$ such that $(\mu_i)^r = 1$ and $\mu_i \neq \mu_j$ if $i \neq j$, and $Q_i \in B(H)$ is idempotent such that $Q_i Q_j = 0$ if $i \neq j$ (cf. [8]).

It follows that μ_i, $1-\mu_i$ and $1-2\mu_i$ is the eigenvalue of $\phi(P)$, $I-\phi(P)$ and $I-2\phi(P)$ respectively. But 1 and 0 are the only two numbers satisfying $\alpha^r = \alpha$, $(1-\alpha)^s = 1-\alpha$ and $(1-2\alpha)^t = 1-2\alpha$, this leads to the idempotence of $\phi(P)$.

Similarly, we can show that ϕ^{-1} also preserves idempotents. Thus ϕ preserves idempotents in both directions. One can complete the proof using Theorem 3.6. □

We now describe the additive mapping preserving square-zero.

Theorem 3.8. *Let H be a complex infinite dimensional Hilbert space, and $\phi : B(H) \to B(H)$ be a unital additive surjective mapping with the property that $(\phi(T))^2 = 0$ if and only if $T^2 = 0$ for every $T \in B(H)$. Then there exists a bounded linear or conjugate-linear bijective mapping $A : H \to H$ such that for every $T \in B(H)$ either $\phi(T) = ATA^{-1}$ or $\phi(T) = AT^{\mathrm{tr}}A^{-1}$.*

Proof. Using the same argument as in the proof of Theorem 2.1 in [12], we can infer that ϕ is an additive bijective mapping preserving idempotents in both directions. Now the theorem goes easily. □

Recall that an operator $T \in B(H)$ is called involutive provided $T^2 = I$. To our knowledge, there is no characterizations of linear or additive involution preservers.

Theorem 3.9. *Let H be a complex infinite dimensional Hilbert space. Let $\phi : B(H) \to B(H)$ be a unital additive surjective mapping satisfying $(\phi(T))^2 = I$ if and only if $T^2 = I$ for every $T \in B(H)$, then there exists a bounded linear or conjugate-linear bijective mapping $A : H \to H$ such that either $\phi(T) = ATA^{-1}$ or $\phi(T) = AT^{\mathrm{tr}}A^{-1}$ for every $T \in B(H)$.*

Proof. We first show the injectivity of ϕ.

If $\phi(T) = 0$, then $\phi(I \pm T) = I$, and so $(I \pm T)^2 = I$, therefore $T = 0$.

We now shall show that ϕ preserves idempotents in both directions.

For arbitrary idempotent $P \in B(H)$, obviously $(I - 2P)^2 = I$. Then $(I - 2\phi(P))^2 = I$, hence $\phi(P)^2 = \phi(P)$. Then ϕ preserves idempotents.

Similarly, ϕ^{-1} also preserves idempotents. Hence ϕ preserves idempotents in both directions. Now applying Theorem 3.6, we can complete the proof. □

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